INTRINSIC SPECTRAL GEOMETRY OF THE KERR-NEWMAN EVENT HORIZON

MARTIN ENGMAN AND RICARDO CORDERO SOTO

Abstract. We uniquely and explicitly reconstruct the instantaneous intrinsic metric of the Kerr-Newman Event Horizon from the spectrum of its Laplacian. In the process we find that the angular momentum parameter, radius, area; and in the uncharged case, mass, can be written in terms of these eigenvalues. In the uncharged case this immediately leads to the unique and explicit determination of the Kerr metric in terms of the spectrum of the event horizon. Robinson's "no hair" theorem now yields the corollary: One can "hear the shape" of noncharged stationary axially symmetric black hole space-times by listening to the vibrational frequencies of its event horizon only.

1. Introduction

Although in general the spectrum of the Laplacian on a manifold determines, via the heat kernel, a sequence of invariants which restrict the geometry, it is a rare occasion indeed that the metric is uniquely determined by the eigenvalues (see, for example, Gordon [6]). In fact one of the few uniqueness results of this kind was proved by Brüning and Heintze, [1]: An S^1 invariant two dimensional surface diffeomorphic to the sphere, which has, in addition, a mirror symmetry about its equator is uniquely determined by its spectrum. This has been generalized by Zelditch, [16], but the Brüning and Heintze result is the appropriate setting for our purposes since this is exactly the class of metrics which include Kerr-Newman event horizons.

In 1973 Smarr [11] studied the metric of the horizon in terms of a scale parameter and a distortion parameter and in a particularly convenient coordinate system for calculating, for example, the curvature. More recently, the first author of the present paper has used a similar coordinate system to study the spectrum of S^1 invariant surfaces. As it turns out, Smarr's form of the horizon metric is a scaled version of a special case of our form of the metric.

S^1 invariant metrics on S^2 can be described in terms of a single function $f(x)$. One can show that the sum of the reciprocals of the nonzero, S^1 invariant eigenvalues (i.e. the trace of the S^1 invariant Greens operator) is given by an integral involving the function $f(x)$. The key to the results we obtain here is that, in the case of the Kerr-Newman event horizon, the function $f(x)$ is a simple rational function and the above mentioned integral can be calculated explicitly. Together with a similar equivariant trace formula, one can now write Smarr's parameters in terms of these traces and hence explictly display the metric in terms of it's spectrum. As far as we know, this is the only nontrivial example of this phenomonen (the Brüning and Heintze result is not constructive in this sense).

2000 *Mathematics Subject Classification.* Primary 58J50; Secondary 83C15, 83C57.

An interesting byproduct of our results (in the uncharged case), together with Robinson's uniqueness theorem, is the unique reconstruction of space-time in terms of these eigenvalues. We conclude the paper with a discussion of possible physical interpretations of these results.

2. Smarr's form of the metric

The *Kerr-Newman metric* describing the geometry of a rotating charged black hole written in Kerr ingoing coordinates (v, r, θ, ϕ) is

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2mr - e^2}{\Sigma}\right) dv^2 + 2drdv - \frac{(2mr - e^2)2a\sin^2\theta}{\Sigma} dvd\phi \\ &-2a\sin^2\theta drd\phi + \Sigma d\theta^2 + \left(\frac{(r^2 + a^2)^2 - \delta a^2 \sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2, \end{aligned} \tag{1}$$

in which (m, a, e) represent respectively, the total mass, angular momentum per unit mass, and the charge. Also $\Sigma = r^2 + a^2\cos^2\theta$ and $\delta = r^2 - 2mr + a^2 + e^2$ (we use the lower case letter here, instead of the traditional upper case, to avoid confusion with the equally traditional use of Δ for the Laplacian). The uncharged $(e = 0)$ case is the *Kerr metric.*

This family of metrics is quite general. To quote Wald [15] ". . . the Kerr-Newman family of solutions completely describes all the stationary black holes which can possibly occur in general relativity." This fact is due to the famous uniquess theorems of Israel [7], Carter and Robinson [10], and Mazur [8] for example.

For the Kerr-Newman metric the surface of the event horizon can be thought of as a spacelike slice through the null hypersurface defined by the largest root, r_+, of $\delta = 0$ i.e. $r_+ = m + \sqrt{m^2 - a^2 - e^2}$. The intrinsic instantaneous metric on the event horizon is obtained by setting $r = r_+$, so that $dr = 0$, and also setting $dv = 0$ to get

$$ds^2_{eh} = (r_+^2 + a^2\cos^2\theta)d\theta^2 + \left(\frac{(r_+^2 + a^2)^2}{r_+^2 + a^2\cos^2\theta}\right)\sin^2\theta d\phi^2 \tag{2}$$

In [11], Smarr defines *the scale parameter* by $\eta = \sqrt{r_+^2 + a^2}$ and *the distortion parameter* by $\beta = \frac{a}{\sqrt{r_+^2 + a^2}}$ and introducing a new variable $x = \cos\theta$ one finds that the event horizon metric is

$$ds^2_{eh} = \eta^2 \left(\frac{1}{f(x)}dx^2 + f(x)d\phi^2\right) \tag{3}$$

where $(x, \phi) \in (-1, 1) \times [0, 2\pi)$ and

$$f(x) = \frac{1 - x^2}{1 - \beta^2(1 - x^2)}. \tag{4}$$

It is well known that the Gauss curvature of a metric in this form is simply $K(x) = -1/(2\eta^2)f''(x)$ so that in this case

$$K(x) = \frac{1}{\eta^2}\left(\frac{1 - \beta^2(1 + 3x^2)}{(1 - \beta^2(1 - x^2))^3}\right), \tag{5}$$

and the surface area of the metric is $A = 4\pi\eta^2$. We point out that in case $a = 0$ ($\beta = 0$), $e = 0$ (1) gives the Schwarzschild black hole and (3) is the metric of the constant Gauss curvature $= \frac{1}{\eta^2}$ metric on S^2.

3. Spectrum of S^1 invariant metrics

For any Riemannian manifold with metric g_{ij} *the Laplacian* is given by

$$\Delta_g = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial}{\partial x^j}\right).$$

This is the Riemannian version of the *Klein-Gordon*, or *D'Alembertian*, or *wave* operator usually denoted by $\Box$.

In this section we outline some our previous work on the spectrum of the Laplacian on S^1 invariant metrics on S^2. The interested reader may consult [2], [3] and [4] for further details.

To simplify the discussion the area of the metric is normalized to $A = 4\pi$ for this section only. The metrics we study have the form:

$$dl^2 = \frac{1}{f(x)}dx^2 + f(x)d\phi^2 \tag{6}$$

where $(x, \phi) \in (-1, 1) \times [0, 2\pi)$ and $f(x)$ satisfies $f(-1) = 0 = f(1)$ and $f'(-1) = 2 = -f'(1)$. In this form, it is easy to see that the Gauss curvature of this metric is given by $K(x) = (-1/2)f^{''}(x)$. The canonical (i.e. constant curvature) metric is obtained by taking $f(x) = 1 - x^2$ and the metric (3) is a homothety (scaling) of a particular example of the general form (6).

The Laplacian for the metric (6) is

$$\Delta_{dl^2} = -\frac{\partial}{\partial x}\left(f(x)\frac{\partial}{\partial x}\right) - \frac{1}{f(x)}\frac{\partial^2}{\partial \phi^2}.$$

Let λ be any eigenvalue of $-\Delta$. We will use the symbols E_λ and $\dim E_\lambda$ to denote the eigenspace for λ and it's multiplicity (degeneracy) respectively. In this paper the symbol λ_m will always mean the mth <u>distinct</u> eigenvalue. We adopt the convention $\lambda_0 = 0$. Since S^1 (parametrized here by $0 \leq \phi < 2\pi$) acts on (M, g) by isometries we can separate variables and because $\dim E_{\lambda_m} \leq 2m + 1$ (see [4] for the proof), the orthogonal decomposition of E_{λ_m} has the special form

$$E_{\lambda_m} = \bigoplus_{k=-m}^{k=m} e^{ik\theta}W_k$$

in which $W_k (= W_{-k})$ is the "eigenspace" (it might contain only 0) of the ordinary differential operator

$$L_k = -\frac{d}{dx}\left(f(x)\frac{d}{dx}\right) + \frac{k^2}{f(x)}$$

with suitable boundary conditions. It should be observed that $\dim W_k \leq 1$, a value of zero for this dimension occuring when λ_m is not in the spectrum of L_k.

The set of positive eigenvalues is given by $Spec(dl^2) = \bigcup_{k\in\mathbb{Z}} SpecL_k$ and consequently the nonzero part of the spectrum of $-\Delta$ can be studied via the spectra $SpecL_k = \{0 < \lambda_k^1 < \lambda_k^2 < \cdots < \lambda_k^j < \cdots\}\forall k \in \mathbb{Z}$. The eigenvalues λ_0^j in the case $k = 0$ above are called the S^1 *invariant eigenvalues* since their eigenfunctions

are invariant under the action of the S^1 isometry group. If $k \neq 0$ the eigenvalues are called k *equivariant* or simply *of type* $k \neq 0$. Each L_k has a Green's operator, $\Gamma_k : (H^0(M))^\perp \to L^2(M)$, whose spectrum is $\{1/\lambda_k^j\}_{j=1}^\infty$, and whose trace is defined by

$$\gamma_k \equiv \sum_{j=1}^{\infty} \frac{1}{\lambda_k^j}. \tag{7}$$

The formulas of present interest were derived in [2] and [3] and are given by

$$\gamma_0 = \frac{1}{2}\int_{-1}^{1} \frac{1-x^2}{f(x)}dx \tag{8}$$

and

$$\gamma_k = \frac{1}{|k|} \qquad \text{if } k \neq 0 \tag{9}$$

Remark. One must be careful with the definition of γ_0 since $\lambda_0^0 = 0$ is an S^1 invariant eigenvalue of $-\Delta$. To avoid this difficulty we studied the S^1 invariant spectrum of the Laplacian on 1-forms in [3] and then observed that the nonzero eigenvalues are the same for functions and 1-forms.

4. Spectral Determination of the Event Horizon

In case $f(x)$ is given by (4) the metric (3) is related to (6) via the homothety $ds_{eh}^2 = \eta^2 dl^2$, and it is well known that

$$\lambda \in Spec(dl^2) \qquad \text{if and only if} \qquad \frac{\lambda}{\eta^2} \in Spec(\eta^2 dl^2)$$

so that, after an elementary integration, the trace formulae for the event horizon are

$$\gamma_0 = \eta^2 \left(1 - \frac{2\beta^2}{3}\right) \tag{10}$$

and

$$\gamma_k = \frac{\eta^2}{|k|} \qquad \forall k \neq 0 \tag{11}$$

For the $k = 1$ case, for example, one can invert the resulting pair of equations to get

$$\beta^2 = \frac{3}{2}\left(1 - \frac{\gamma_0}{\gamma_1}\right) \tag{12}$$

and

$$\eta^2 = \gamma_1, \tag{13}$$

and thereby write the event horizon metric explicitly and uniquely in terms of the spectrum as follows.

Proposition. *With γ_0 and γ_1 defined as in* (7) *the instantaneous intrinsic metric of the Kerr-Newman event horizon is given by*

$$ds_{eh}^2 = \gamma_1 \left(\frac{1 - \frac{3}{2}\left(1 - \frac{\gamma_0}{\gamma_1}\right)(1-x^2)}{1-x^2} dx^2 + \frac{1-x^2}{1 - \frac{3}{2}\left(1 - \frac{\gamma_0}{\gamma_1}\right)(1-x^2)} d\phi^2 \right). \tag{14}$$

An immediate consequence of (11) is that the area of the metric has a representation for each $k \in \mathbb{N}$ given by

$$A = 4\pi k \gamma_k. \tag{15}$$

Remarks. i.) One might view equations Eqs. (10) and (11) as nothing more than definitions of new parameters in terms of the old. On the other hand, the quantities γ_k are fundamental and naturally defined quantities coming from the discrete set of vibrational wave frequencies on the surface. Alternatively, one can think of each trace, γ_k as the sum of the squares of all wavelengths of given quantum number k.

ii.) One can use γ_0 together with γ_k for any k to reconstruct the metric. In some sense one can "construct the metric in $\aleph_0$ ways".

iii.) Brüning and Heintze proved that for S^1 invariant metrics symmetric about the equator the S^1 invariant spectrum determines the metric. Their result requires the prescription of all of the eigenvalues of the $k = 0$ spectrum to uniquely determine the surface of revolution. In the example of the present paper the metric is, therefore, uniquely determined once we have knowledge of the entire list of S^1 invariant eigenvalues. For the explicit construction one exchanges complete knowledge of the $k = 0$ spectrum for partial spectral data, namely the traces of the Greens operators for $k = 0$ and any $k \neq 0$.

All the physical parameters can also be written in terms of the spectra as follows:

$$a^2 = \frac{3}{2}(\gamma_1 - \gamma_0) \tag{16}$$

$$r_+^2 = \frac{3\gamma_0 - \gamma_1}{2} \tag{17}$$

$$m^2 = \frac{(\gamma_1 + e^2)^2}{6\gamma_0 - 2\gamma_1}, \tag{18}$$

as well as the angular velocity and surface gravity. We observe, however, that the mass depends on the charge e as well as the spectrum so that the mass is uniquely determined by the spectrum only in the Kerr ($e = 0$) case.

We set together Remark iii. with (16), (18) (with $e = 0$), (1), and Robinson's uniqueness Theorem [10] to observe.

Theorem. *A noncharged stationary axially symmetric asymptotically flat vacuum space-time with a regular event-horizon is uniquely determined by the S^1 invariant spectrum of the intrinsically defined Laplacian on its event horizon. The space-time is explicitly constructed from the S^1 invariant trace together with any $k \neq 0$ trace of the associated Green's operator on the event horizon.*

5. Discussion

It is well known (see [5], [13], [9] among many others) that the quasinormal mode frequencies for scalar, electromagnetic, and gravitational radiation of black holes are related to the separation constants arising from separating variables in the Teukolsky master equation. These "characteristic sounds" of the black hole are considered to be observable in an astrophysical sense. The angular operator (and, respectively, its separation constants) coming from the separation of variables is closely related to the Laplacian (and, respectively, its eigenvalues) on the event horizon in the sense that, for scalar fields, they both reduce to the Laplacian (and

corresponding eigenvalues) on the constant curvature S^2 for the Schwarzschild ($a = 0$) case. There is also some evidence (this is a work in progress) that the $a > 0$ Teukolsky angular equation can be viewed, after a change of variables, as a Laplace eigenvalue equation for a fourth order Taylor polynomial approximation of the horizon metric function. We hope to pursue these matters in our future work.

On a more philosophical note, the reader may have noticed that the main result of this paper is consistent with the holographic principle ([12], [14]) in as much as the structure of the ($3+1$ dimensional) Kerr-Newman space-time is encoded in the intrinsic spectral data of the (two-dimensional) event horizon surface.

These phenomena suggest that the spectrum of the Laplacian on event horizons is playing an important, if rather subtle and not well understood, role in the physics of the space-time manifold.

6. Acknowledgements

The authors thank Thomas Powell, Steve Zelditch, and Roberto Ramírez for their ideas and assistance in the preparation of this paper. This work was partially supported by the NSF Grant: Model Institutes for Excellence at UMET.

References

[1] Brüning, J., Heintze, E. *Spektrale Starrheit gewisser Drehflächen*, Math. Ann., **269**, 1992, 95-101.
[2] Engman, M., *New Spectral Characterization Theorems for S^2*, Pacific J. Math. Vol. 154, No. 2, 1992, 215-229.
[3] Engman, M., *Trace Formulae for S^1 invariant Green's Operators on S^2*, manuscripta math. 93, (1997), 357-368.
[4] Engman, M., *Sharp bounds for eigenvalues and multiplicities on surfaces of revolution*, Pacific J. Math. Vol. 186, No. 1, 1998, 29-37.
[5] Frolov, V., Novikov, I, *Black Hole Physics: Basic Concepts and New Developments*, Kluwer Academic Publishers, Dordrecht, 1998.
[6] Gordon, C., *Handbook of differential geometry*, Vol. I, 747-778, North-Holland, Amsterdam, 2000.
[7] Israel, W., *Phys. Rev.* **164**, 1776-9 (1968).
[8] Mazur, P., *Proof of Uniqueness of the Kerr-Newman Black Hole Solution*, J. Phys. A: Math. Gen. **15**, 3173-3180 (1982).
[9] Nollert, H., *Quasinormal modes: the characteristic 'sound' of black holes and neutron stars*, *Class. Quantum. Grav.* **16**, R 159 (1999).
[10] Robinson, D.C., *Uniqueness of the Kerr Black Hole*, Phys. Rev. Lett. **34**, 905.
[11] Smarr, L., *Surface Geometry of Charged Rotating Black Holes*, Phys. Rev. **D7** 289, (1973).
[12] Susskind, L., "The World as a Hologram", *J. Math. Phys.* **36**, 6377-6396, (1995).
[13] Teukolsky, S., *Rotating black holes: Separable wave equations for gravitational and electromagnetic perturbations*, *Phys. Rev. Lett.* **29**, 1114 (1972).
[14] 't Hooft, G., *Dimensional Reduction in Quantum Gravity*, in Salamfestschrift: a collection of talks, World Scientific Series in 20th Century Physics Vol. 4, edited by A. Ali, J. Ellis, and S. Randjbar-Daemi (World Scientific 1993).
[15] Wald, R., *Space, Time, and Gravity: The Theory of the Big Bang and Black Holes, 2nd Ed.*, The University of Chicago Press, Chicago, 1992.
[16] Zelditch, S., *The inverse spectral problem for surfaces of revolution* , J. Differential. Geom. 49 (1998), no. 2, 207-264.

Departamento de Ciencias y Tecnología, Universidad Metropolitana, San Juan, PR 00928

E-mail address: um_mengman@suagm.edu

E-mail address: mathengman@yahoo.com